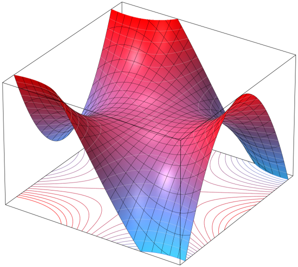
# Perturbing the monkey's saddle

The **monkey saddle** $\Re(z^3) = x^3 - 3xy^2$ is not Morse:



- How many different natural and interesting ways can you think of to perturb it to be Morse?
- How are these related by generic homotopies of functions? What kinds of singularities do you encounter?
- How are these homotopies related by generic homotopies of homotopies? What kinds of singularities appear now?

# Indefinite Morse 2–functions

## (and some reflections on the monkey saddle)

David Gay[1] Rob Kirby[2]

[1]Euclid Lab

[2]U.C. Berkeley

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# Morse functions and Cerf theory

- **Morse function:** Generic $g\colon M^m \to Y^1$, $g(\partial M) \subset \partial Y$
  - Usual local models
  - Isolated critical points, isolated critical values
- **Cerf theory:** Generic homotopy $g_t\colon M^m \to Y^1$ between Morse functions $g_0$, $g_1$ is Morse except at discrete values $t = t_*$
  - Local models when $g_{t_*}$ not Morse: critical values cross, births/deaths
  - **Cerf graphic:** (visualization tool) image of 1–dimensional singular locus of $(t, p) \mapsto (t, g_t(p))$, $I \times M \to I \times Y$
- **Two parameters:** Generic homotopy $g_{s,t}$ between generic homotopies $g_{0,t}$ and $g_{1,t}$ between Morse functions is a generic homotopy between Morse functions except at discrete values $s = s_*$
  - Local models when $g_{s_*,t}$ not generic: well understood, more later
- **Indefinite:** no minimal or maximal index critical points

# Morse 2–functions

- **Morse 2–function:** Generic map $G\colon X^n \to \Sigma^2$, $G(\partial X) \subset \partial\Sigma$
  - Example: $X = I \times M^{n-1}$, $\Sigma = I \times Y^1$, $G(t,p) = (t, g_t(p))$ for generic $g_t\colon M^{n-1} \to Y^1$
  - Locally all Morse 2–functions look like this example
- **"Cerf theory" for Morse 2–functions:** Study generic homotopies $G_s\colon X^n \to \Sigma^2$ between Morse 2–functions $G_0$ and $G_1$
  - Example: $X = I \times M^{n-1}$, $\Sigma = I \times Y^1$, $G_s(t,p) = (t, g_{s,t}(p))$ for generic homotopy of homotopies $g_{s,t}\colon M^{n-1} \to Y^1$
  - Locally all generic homotopies between Morse 2–functions look like this example
  - In particular, $G_s$ is a Morse 2–function except at discrete values $s = s_*$
- **Indefinite:** no minimal or maximal index critical points in local models

# Indefinite $S^2$–valued Morse 2–functions

## Theorem

- ***Existence*** *($n \geq 3$): Given*
  - *Closed (connected) $X^n$*
  - *Framed (connected) $F^{n-2} \subset X$*

  *Then*
  - $\exists$ indefinite *Morse 2–function $G\colon X \to S^2$ with $G^{-1}(\text{n.p.}) = F^{n-2}$ (framed) (*Saeki*)*
  - *(All fibers of G are connected)*
- ***Uniqueness*** *($n \geq 4$): Given*
  - *$G_0, G_1 \colon X \to S^2$ homotopic, indefinite Morse 2–functions*
  - *(All fibers of $G_0$ and $G_1$ connected)*

  *Then:*
  - $\exists$ indefinite *generic homotopy $G_s \colon X \to S^2$ between $G_0$ and $G_1$ (*Williams*)*
  - *(All fibers of $G_s$ are connected for all s)*

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# Indefinite $D^2$–valued Morse 2–functions

**Theorem**

- ***Existence*** *($n \geq 3$): Given*
  - *Compact, connected $X^n$, $\partial X \neq \emptyset$*
  - *Indefinite Morse $g\colon \partial X \to S^1$ (with connected level sets)*

  *Then*
  - $\exists$ indefinite *Morse 2–function $G\colon X \to D^2$ with $G|_{\partial X} = g$*
  - *(All fibers of $G$ are connected)*
- ***Uniqueness*** *($n \geq 4$): Given*
  - *$X$, $g$ as above*
  - *$G_0, G_1\colon X \to D^2$ indefinite Morse 2–functions with $G_0|_{\partial X} = G_1|_{\partial X} = g$*
  - *(All fibers of $G_0$ and $G_1$ connected)*

  *Then:*
  - $\exists$ indefinite *generic homotopy $G_s\colon X \to D^2$ between $G_0$ and $G_1$, with $G_s|_{\partial X} = g$ for all $s$*
  - *(All fibers of $G_s$ are connected for all $s$)*

# Indefinite $D^2$–valued Morse 2–functions

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- ***Existence*** *($n \geq 3$): Given*
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  - *(All fibers of $G_0$ and $G_1$ connected)*

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  - *(All fibers of $G_s$ are connected for all $s$)*

Coordinates on $I^2 = I \times I$: $(t, z)$

**Theorem**

- ***Existence*** *($n \geq 3$): Given*
  - *Closed $F_{ij}^{n-2}$ for $i, j \in \{0, 1\}$, $F_{0j} \cong F_{1j}$*
  - *Cobordisms $M_i^{n-1}$ from $F_{i0}$ to $F_{i1}$, $i = 0, 1$*
  - *Indefinite Morse $g_i \colon M_i \to I$, $i = 0, 1$, (with connected level sets)*
  - *Connected cobordism-with-sides $X^n$ from $M_0$ to $M_1$, i.e. $\partial X_n = -M_0 \cup (I \times F_{00}) \cup (I \times (-F_{01})) \cup M_1$*

*Then*

  - $\exists$ indefinite *Morse 2–function $G \colon X \to I^2$ such that:*
  - $G(M_i) = \{i\} \times I$
  - $z \circ G|_{M_i} = g_i$
  - *$t \circ G|_{I \times (\pm F_{0i})}$ is projection $I \times (\pm F_{0i}) \to I$*
  - *(All fibers of G are connected)*

# Indefinite $I^2$–valued Morse 2–functions continued

**Theorem**

- ***Uniqueness*** $(n \geq 4)$: *Given*
  - $X$, $M_i$, $F_{ij}$, $g_i$ *as above*
  - $G_0, G_1 : X \to I^2$ *satisfying conditions satisfied by G above*
  - *(All fibers of* $G_0$ *and* $G_1$ *connected)*

*Then:*

  - $\exists$ indefinite *generic homotopy* $G_s : X \to I^2$ *from* $G_0$ *to* $G_1$*, with* $G_s|_{\partial X} = G_0|_{\partial X}$ *for all s*
  - *(All fibers of* $G_s$ *connected for all s)*

- **Easy:** $I^2$–valued existence $\implies$ $D^2$–valued existence $\implies$ $S^2$–valued existence.
- **Easy:** $I^2$–valued uniqueness $\implies$ $D^2$–valued uniqueness.
- **Not totally trivial:** $D^2$–valued uniqueness $\implies$ $S^2$–valued uniqueness.

# Morse and Cerf theory ingredients:

**Theorem**

*Given connected cobordism $M^m$ from closed, nonempty (connected) $F_0^{m-1}$ to $F_1^{m-1}$, with framed submanifold $K^k \subset M$, $k \leq m-2$:*

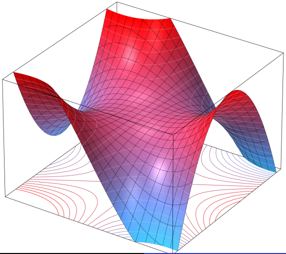
- ***Existence:*** *$(m \geq 1)$* $\exists$ indefinite *Morse function $g\colon M \to I$ with $g(F_j) = j$, with $K$ framed in level set(s).*
- ***Uniqueness:*** *$(m \geq 2)$ Given indefinite Morse functions $g_0, g_1\colon M \to I$ with $g_i(F_j) = j$, with $K$ framed in level set(s),* $\exists$ indefinite *generic homotopy $g_t\colon M \to I$ between $g_0$ and $g_1$* (Kirby) *keeping $K$ framed in the level set(s).*
- ***Two-parameter uniqueness:*** *$(m \geq 3)$ Given $g_0$, $g_1$ as above, and two indefinite generic homotopies $g_{0,t}, g_{1,t}\colon M \to I$ between $g_0 = g_{0,0} = g_{1,0}$ and $g_1 = g_{0,1} = g_{1,1}$,* $\exists$ indefinite *generic homotopy $g_{s,t}\colon M \to I$ between $g_{0,t}$ and $g_{1,t}$, with $g_{s,0} = g_0$ and $g_{s,1} = g_1$ for all $s$.*

# Ideas of proofs

**Proof.**

- **Existence:** Get $K$ in level set(s), cancel 0–handles avoiding $K$
- **Uniqueness:** Cancel 0–handles and swallowtails avoiding $K$
- **Two-parameter uniqueness:** Cancel 0–handles and swallowtails, then *flip monkey saddles*.

□



# Morse 2–function building blocks:

Existence of indefinite $I^2$–valued Morse 2–functions on $X^n$ follows from:

- Existence for indefinite Morse functions on $X^n$
- Uniqueness for indefinite Morse functions on $M^{n-1}$
- The following theorem:

**Theorem**

*Given $F_{ij}^{n-2}$, $M_i^{n-1}$, $X^n$, with:*

- *Morse function $\tau\colon X \to I$ with exactly one critical point of index $\leq n-2$, with framed attaching sphere $K \subset M_0$*
- *Morse function $g\colon M_0 \to I$ with $K$ framed in a level set*

*Then $\exists$ Morse 2–function $G\colon X \to I$ with:*

- $t \circ G = \tau$
- $z \circ G|_{M_0} = g_0$

# Ingredients for indefinite Morse 2–function uniqueness:

Given $F_{ij}$, $M_i$, $X$, indefinite $G_0, G_1 : X \to I^2$ which agree on $M_i$

**Theorem**

*If $t \circ G_0 = t \circ G_1$ then* $\exists$ indefinite *generic homotopy $G_s$ between $G_0$ and $G_1$ with $t \circ G_s = t \circ G_0$ for all $s$.*

**Theorem**

*Let $\tau_0 = t \circ G_0, \tau_1 = t \circ G_1 : X \to I$. Given an indefinite generic homotopy $\tau_s$ between $\tau_0$ and $\tau_1$, $\exists$ an indefinite generic homotopy $G'_s$ between $G'_0 = G_0$ and $G'_1$ with $t \circ G'_s = \tau_s$.*