Gay, D. T. and A. I. Stipsicz. (2007) "Symplectic Rational Blow-down Along Seifert Fibered 3-Manifolds," International Mathematics Research Notices, Vol. 2007, Article ID rnm084, 20 pages.
doi:10.1093/imrn/rnm084

Symplectic Rational Blow-down Along Seifert Fibered 3-Manifolds

David T. Gay[1] and András I. Stipsicz[2]

[1]Department of Mathematics and Applied Mathematics, University of Cape Town, Private Bag X3, Rondebosch 7701, South Africa, and [2]Rényi Institute of Mathematics, Hungarian Academy of Sciences, Reáltanoda utca 13-15, H-1053 Budapest, Hungary

Correspondence to be sent to: András I. Stipsicz, Rényi Institute of Mathematics, Hungarian Academy of Sciences, Reáltanoda utca 13-15, H-1053 Budapest, Hungary.
e-mail: stipsicz@math-inst.hu

We verify that the rational blow down schemes along certain Seifert fibered 3-manifolds found by Stipsicz, Szabó and Wahl are, in fact, symplectic operations.

1 Introduction

The rational blow-down procedure (introduced by Fintushel and Stern [4] and generalized by Park [25]) turned out to be one of the most effective operations in constructing smooth 4-manifolds with interesting topological properties, cf. [5, 26–28]. Recall that when performing the rational blow-down operation we simply replace the tubular neighbourhood of a string of 2-spheres (intersecting each other according to the linear plumbing tree with framings specified by the continued fraction coefficients of $-\frac{p^2}{pq-1}$ for some p, q relatively prime) with a rational homology disk. The success of this operation might be explained by the fact that—as Symington showed [30, 31]—it can be performed symplectically. More precisely, if the ambient 4-manifold is symplectic and the spheres are symplectic submanifolds intersecting each other orthogonally then the neighbourhood

Received July 10, 2007; Revised July 25, 2007; Accepted August 10, 2007
Communicated by Yasha Eliahberg

See http://www.oxfordjournals.org/our_journals/imrn/ for proper citation instructions.

can be chosen so that the symplectic structure (when restricted to the complement of this neighbourhood) extends over the glued-in rational homology disk. Symington's argument used a beautiful application of toric geometry by constructing both the right symplectic neighbourhood and the appropriate symplectic model for the rational homology disk.

It is not hard to list the combinatorial constraints a plumbing tree must satisfy in order for the proofs of Fintushel-Stern and Park to be applied. In [29] these constraints were explicitly spelled out and the family of plumbing trees satisfying these properties has been determined. The combinatorial conditions are, however, sometimes too weak to ensure the existence of the rational homology disk needed to perform the geometric operation. In [29] a number of examples were shown to admit such rational homology disks. (Some of these examples were already applied by Michalogiorgaki [16] for constructing exotic 4-manifolds.) Whether these new examples for rational blow-down can be executed in the symplectic category, however, remained an open question.

In this paper we show that some of the plumbing trees found in [29] can be, in fact, blown down symplectically. (For the definition of *symplectic* rational blow-down see also [30, Definition 1.1].) Our main result refers to a class of plumbing trees which we describe in the following definition. However, it is worth pointing out here that we are able to prove the existence of ω-convex neighborhoods (which amounts to half of our main result) for a much larger class of plumbing trees, namely all star-shaped negative definite trees.

Definition 1.1.

- The plumbing tree given by Figure 1 will be denoted by $\Gamma_{p,q,r}$ ($p,q,r \geq 0$). We denote the set of these plumbing trees with $\mathcal{W}$.

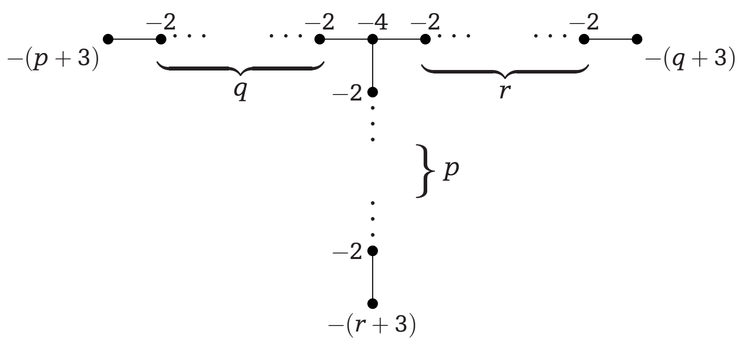

Figure 1 The graph $\Gamma_{p,q,r}$ in $\mathcal{W}$.

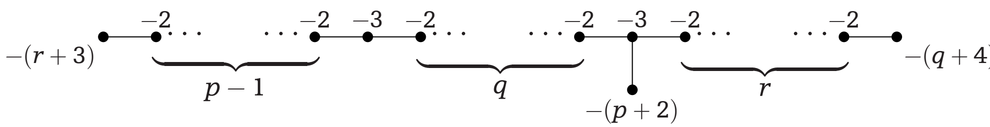

Figure 2 The graph $\Delta_{p,q,r}$ for $p \geq 1$ and $q, r \geq 0$.

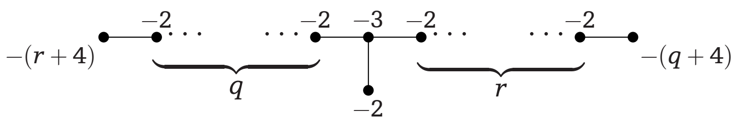

Figure 3 The graph $\Delta_{0,q,r}$.

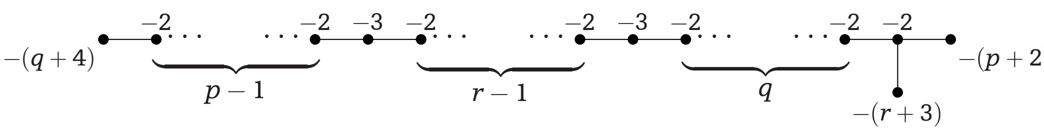

Figure 4 The graph $\Lambda_{p,q,r}$ for $p, r \geq 1$ and $q \geq 0$.

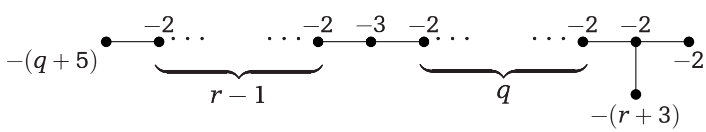

Figure 5 The graph $\Lambda_{0,q,r}$ for $r \geq 1$ and $q \geq 0$.

- The plumbing tree of Figure 2 will be denoted by $\Delta_{p,q,r}$ ($p \geq 1$ and $q, r \geq 0$). The slight modification of the graph $\Delta_{p,q,r}$ when $p = 0$ is shown in Figure 3. The set of graphs $\Delta_{p,q,r}$ with $p, q, r \geq 0$ will be denoted by $\mathcal{N}$.
- The plumbing graph of Figure 4 is $\Lambda_{p,q,r}$ with $p, r \geq 1$ and $q \geq 0$. Modifications of these graphs for $p = 0, r \geq 1$ and $p \geq 1, r = 0$ and finally for $p = r = 0$ are shown by Figures 5, 6 and 7. The set of graphs $\Lambda_{p,q,r}$ with $p, q, r \geq 0$ will be denoted by $\mathcal{M}$.
- Finally let us denote the union $\mathcal{W} \cup \mathcal{N} \cup \mathcal{M}$ simply by $\mathcal{G}$.

Theorem 1.2. Suppose that (X, ω) is a given symplectic 4-manifold and the symplectic spheres $S_i \subset (X, \omega)$ intersect each other ω-orthogonally and according to the plumbing tree Γ, which is an element of $\mathcal{G}$. Then there is a rational homology disk B_Γ and a tubular neighbourhood S_Γ of $\cup S_i \subset X$ such that the symplectic form ω extends from $X - \text{int}S_\Gamma$ to $X_\Gamma = (X - \text{int}S_\Gamma) \cup B_\Gamma$. In short, graphs in $\mathcal{G}$ can be symplectically blown down. $\square$

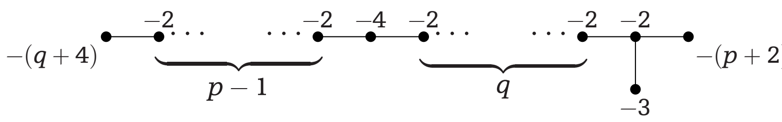

Figure 6 The graph $\Lambda_{p,q,0}$ for $p \geq 1$ and $q \geq 0$.

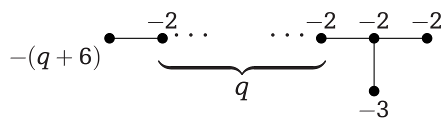

Figure 7 The graph $\Lambda_{0,q,0}$ for $q \geq 0$.

The proof relies on the gluing theorem of symplectic manifolds along ω-convex (and ω-concave) boundaries, as explained in [3]. (Recall that "ω-convex", resp. "ω-concave", is synonymous with "strongly convex", resp. "strongly concave".) According to this scheme, a strong convex symplectic filling of the contact 3-manifold (M_1, ξ_1) can be symplectically glued to a strong concave symplectic filling of (M_2, ξ_2) provided there is a contactomorphism between (M_1, ξ_1) and (M_2, ξ_2). Therefore one way of proving Theorem 1.2 is to show that

(1) the symplectic spheres admit a neighbourhood such that the complement of it is a strong concave filling of the boundary contact 3-manifold,
(2) the rational homology disk B_Γ admits a symplectic structure which is a strong convex filling of its boundary (equipped with some contact structure), and finally
(3) the desired gluing contactomorphism exists.

There are many possibilities for verifying (1) above. Since the plumbing graph under examination is negative definite, a classical theorem of Grauert [12] shows the existence of a J-convex neighbourhood of the spheres, which in this dimension shows that the complement is a weak concave filling of its boundary. The concave analogue of Eliashberg's deformation argument [2] (stating that a weak convex filling of a rational homology 3-sphere can be perturbed to be a strong filling, cf. [19]) would complete the argument—but no concave analogue of the "convex" theorem of Eliashberg mentioned above is available at the moment. We will rather use an explicit way of constructing symplectic structures on the plumbing 4-manifold M_Γ (associated to the plumbing graph Γ) and invoke standard neighbourhood theorems to find the right ω-convex neighborhoods.

For (2) we only need to note that the rational homology disks for the plumbing graphs of $\mathcal{G}$ are given as deformations of appropriate surface singularities (see [29]), therefore are equipped with Stein structures. (Alternatively, B_Γ can be given by Weinstein handle attachments, according to the Kirby diagram described in [29].)

Finally, the existence of the desired contactomorphism (listed under (3) above) will be shown by using the classification of tight contact structures on the boundary 3-manifolds.

2 Convex neighborhoods of certain plumbings

We prove a slightly stronger result than is needed in the proof of our main result. More precisely, we will consider a more general class of graphs than $\mathcal{G}$.

Theorem 2.1. Suppose that (X, ω) is a given symplectic 4-manifold, and the symplectic spheres S_i intersect each other ω-orthogonally and according to a star-shaped graph Γ with negative definite intersection form. Then $\cup S_i$ admits an ω-convex neighbourhood S_Γ. □

The proof of Theorem 2.1 will easily follow from the main result of this section below. (As customary, we will denote the positive half line $(0, \infty)$ by $\mathbb{R}^+$.)

Theorem 2.2. Suppose that Γ is a star-shaped negative definite graph and $a_i \in \mathbb{R}^+$ $(i = 1, \ldots, |\Gamma|)$ are given. Then there is a symplectic structure $\omega_{\mathbf{a}}$ on the plumbing 4-manifold M_Γ associated to the graph $\Gamma \in \mathcal{G}$ such that

- the spheres P_i corresponding to the vertices of the plumbing graph are symplectic and ω-orthogonal,
- $\int_{P_i} \omega_{\mathbf{a}} = a_i$ and
- any neighbourhood of $\cup P_i$ contains an $\omega_{\mathbf{a}}$-convex neighbourhood. □

Proof of Theorem 2.1 from Theorem 2.2. Suppose that (X, ω) is given with $S_i \subset X$ intersecting each other according to Γ and take $a_i = \int_{S_i} \omega$. Consider the plumbing 4-manifold M_Γ corresponding to Γ and equip M_Γ with the symplectic structure $\omega_{\mathbf{a}}$ provided by Theorem 2.2. By Moser's Theorem $\cup S_i \subset (X, \omega)$ and $\cup P_i \subset (M_\Gamma, \omega_{\mathbf{a}})$ admit symplectomorphic neighborhoods (cf. also [30, Proposition 3.5]), which, by the third conclusion of Theorem 2.2 concludes the proof. ■

The idea of the proof of Theorem 2.2 is to construct the desired neighbourhood for each of the legs independently, using toric techniques as in [30], then to construct the desired neighbourhood for the central vertex, and then to glue the pieces together carefully. These techniques will certainly generalize to larger classes of graphs, but here we focus on star-shaped graphs. Also, there should be a way to state these results in terms of embedded graphs in $\mathbb{R}^3$ with edges of rational slope, inspired by the "tropical" ideas of Mikhalkin, but in these simple cases it is easier to avoid this perspective.

It is convenient to work with 5-tuples of the form (X, ω, C, f, V) where:

(1) X is a 4-manifold,
(2) ω is a symplectic form on X,
(3) C is a collection of symplectically embedded surfaces in X intersecting each other ω-orthogonally,
(4) f is a smooth function $f\colon X \to [0, \infty)$ such that $f^{-1}(0) = C$ and f has no critical values in $(0, \infty)$, and
(5) V is a Liouville vector field defined on $X \setminus C$ such that $df(V) > 0$.

If we can produce such a 5-tuple with C being a collection of spheres intersecting according to the given graph Γ with symplectic areas a_i, then we will have proved Theorem 2.2. In our proof we will construct one such 5-tuple for each leg of Γ and one for the central vertex, and then glue them together.

Let us call such a 5-tuple a "neighbourhood 5-tuple". Theorem 2.2 follows from a much more technical statement:

Proposition 2.3. Suppose that, for some $m \in \mathbb{N}$ and some $n_1, \ldots, n_m \in \mathbb{N}$, we are given points in the upper half plane $\{P_{i,j} = (x_{i,j}, y_{i,j}) \in \mathbb{R}^2, y_{i,j} > 0 \mid 1 \leq i \leq m, 0 \leq j \leq n_i + 2\}$, with $P_{i,j} \neq P_{i,j+1}$. For each i, j with $1 \leq i \leq m$ and $0 \leq j \leq n_i + 1$, let $L_{i,j}$ be the line containing $P_{i,j}$ and $P_{i,j+1}$ and let $E_{i,j}$ be the compact oriented line segment from $P_{i,j}$ to $P_{i,j+1}$. Now require that each $L_{i,j}$ has rational or infinite slope, so that there is a well-defined primitive integral tangent vector $\tau_{i,j} = (u_{i,j}, v_{i,j})^T \in \mathbb{Z}^2$ parallel to $E_{i,j}$ pointing from $P_{i,j}$ to $P_{i,j+1}$. (Here "primitive" means that $\gcd(u_{i,j}, v_{i,j}) = 1$ with $\gcd(1, 0)$ defined to be 1.) Record the positive real numbers $\lambda_{i,j}$ such that the vector from $P_{i,j}$ to $P_{i,j+1}$ is equal to $\lambda_{i,j}\tau_{i,j}$. Suppose there is a fixed constant $y_0 > 0$ such that $y_{i,0} = y_0$ for all $i = 1, \ldots, m$ and suppose that $x_{1,0} + x_{2,0} + \cdots + x_{m,0} > 0$. Suppose furthermore that the following conditions are satisfied for each $i = 1, \ldots, m$:

(1) $\tau_{i,0} = (1, 0)^T$; i.e. $x_{i,0} < x_{i,1}$ and $y_{i,0} = y_{i,1}$.
(2) L_{i,n_i+1} passes through the origin $O = (0, 0)$; i.e. $(x_{i,n_i+2}, y_{i,n_i+2}) = \rho(x_{i,n_i+1}, y_{i,n_i+1})$ for some $\rho \in \mathbb{R}^+$.

(3) For each $j = 0, \ldots, n_i$, $\det(\tau_{i,j}, \tau_{i,j+1}) = +1$.
(4) For each $j = 0, \ldots, n_i$, $\det(\tau_{i,j}, P_{i,j}^T) > 0$.

Then there exists a neighbourhood 5-tuple (X, ω, C, f, V) such that C is a collection of symplectically embedded 2-spheres $S_0, S_{i,j}, 1 \leq i \leq m, 1 \leq j \leq n_i$ intersecting ω-orthogonally according to an m-legged star-shaped graph with S_0 the central vertex intersecting each $S_{i,1}$ and with $S_{i,1}, \ldots, S_{i,n_i}$ the i'th leg. Furthermore, the areas and self-intersections are given as follows:

(1) For each $i = 1, \ldots, m$ and each $j = 1, \ldots, n_i$, $S_{i,j} \cdot S_{i,j} = -\det(\tau_{i,j-1}, \tau_{i,j+1})$, and the area of $S_{i,j}$ is $2\pi\lambda_{i,j}$.
(2) Noting that, for each $i = 1, \ldots, m$, $\tau_{i,1} = (u_{i,1}, 1)^T$, we have that $S_0 \cdot S_0 = u_{1,1} + u_{2,1} + \cdots + u_{m,1}$.
(3) The area of S_0 is $2\pi(x_{1,1} + x_{2,1} + \cdots + x_{m,1})$. □

Proof. Throughout this proof, identify S^1 with $\mathbb{R}/2\pi\mathbb{Z}$ through the $\mathbb{R}/2\pi\mathbb{Z}$-valued coordinate function q (possibly decorated with subscripts); i.e. $\int_{S^1} dq = 2\pi$.

To build the neighbourhood 5-tuples for the legs, we give a detailed description of the toric setup as we need it and the basic results that come from the techniques in [30, 31]. (We are essentially ignoring the global group action point of view that is fundamental to traditional toric geometry, and taking the more local topological point of view developed by Symington. Also, here we make essential use of the correspondence between radial vector fields on images of moment maps and Liouville vector fields on the domains of moment maps; this is not standard in the toric geometry literature but is critical to Symington's work and ours.)

For each $i = 1, \ldots, m$, the following is the construction of a neighbourhood 5-tuple $(X_i, \omega_i, C_i, f_i, V_i)$ for the i'th leg. Here C_i will be $D_{i,0} \cup S_{i,1} \cup S_{i,2} \cup \cdots \cup S_{i,n}$ where $D_{i,0}$ is a small open disk neighbourhood in S_0 of $S_0 \cap S_{i,1}$. An example is illustrated in Figure 8.

For each $j = 0, \ldots, n_i + 1$, let $H_{i,j}$ be the closed half-plane with $\partial H_{i,j} = L_{i,j}$ and not containing the origin O, and consider the convex region $H_i = H_{i,0} \cap \cdots \cap H_{i,n_i+1}$. Choose a smooth function $g_i \colon H_i \to [0, \infty)$ with no critical values in $(0, \infty)$ such that:

(1) $g_i^{-1}(0) = (L_{i,0} \cup L_{i,1} \cup \cdots \cup L_{i,n_i}) \cap H_i$.
(2) For some small $\epsilon > 0$ (smaller than $x_{i,1} - x_{i,0}$), g_i restricted to $(-\infty, x_{i,0} + \epsilon] \times [y_{i,0}, y_{i,0} + \epsilon]$ is given by $g_i(x, y) = y - y_{i,0}$.
(3) On a small neighbourhood of $L_{i,n_i+1} \cap H_i$, $g_i(x, y)$ is equal to the Euclidean distance from (x, y) to L_{i,n_i}.

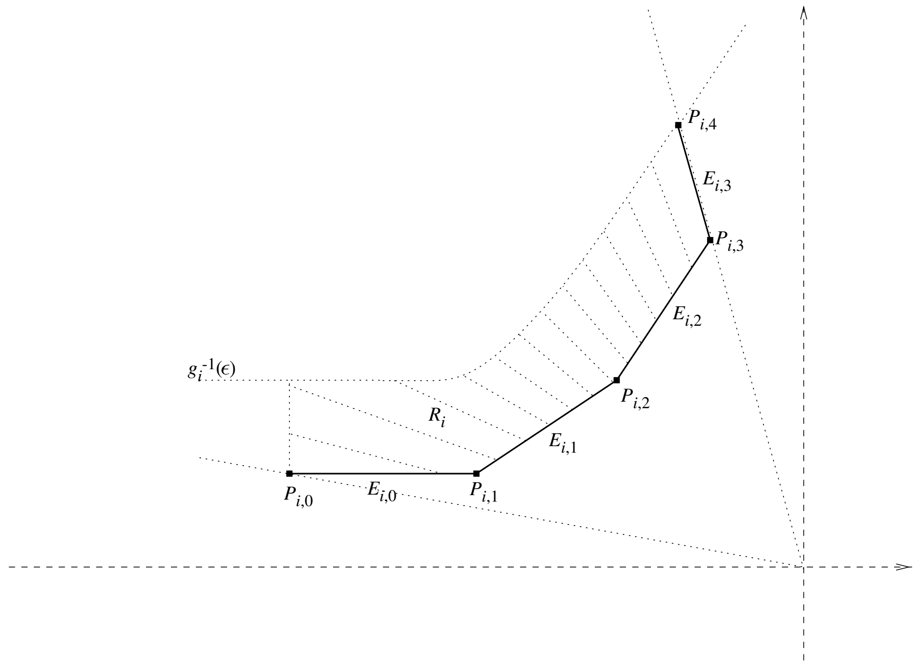

Figure 8 An example of a two-dimensional "moment-map image" of the neighbourhood for one leg of a star-shaped plumbing graph. Here $n_i = 2$.

(4) The level sets of g_i are transverse to the radial vector field radiating out from the origin; i.e. $dg_i(x\partial_x + y\partial_y) > 0$ whenever $dg_i \neq 0$.

Finally consider the region $R_i = g_i^{-1}[0,\epsilon) \setminus ((-\infty, x_{i,0}] \times \mathbb{R})$, and relabel things so that $E_{i,0} \cap R_i$ is now called $E_{i,0}$ and $E_{i,n_i+1} \cap R_i$ is now called E_{i,n_i+1}. (These are the initial and final edges of the polygonal boundary of R_i, and they are bounded but noncompact, while the other edges $E_{i,1}, \dots, E_{i,n}$ are compact.) Also let P_{i,n_i+2} now denote the new terminal point of E_{i,n_i+1}, so that E_{i,n_i+1} is the half-open interval from P_{i,n_i+1} to P_{i,n_i+2}, containing P_{i,n_i+1} and not P_{i,n_i+2}. This 2-dimensional data is the "template" for our neighbourhood 5-tuple, which we now describe in some detail.

Let $Y_i = \{(p_1, q_1, p_2, q_2) \in \mathbb{R} \times S^1 \times \mathbb{R} \times S^1 \mid (p_1, p_2) \in R_i\}$ with symplectic form $\eta_i = dp_1 \wedge dq_1 + dp_2 \wedge dq_2$ and let $\pi_i \colon Y_i \to R_i$ be the projection $(p_1, q_1, p_2, q_2) \mapsto (p_1, p_2)$. For each edge $E_{i,j}$ let $Y_{i,j} = [0, \lambda_{i,j}] \times S^1$ with coordinates (p, q) and symplectic form $\eta_{i,j} = dp \wedge dq$ and let $\pi_{i,j} \colon Y_{i,j} \to E_{i,j}$ be the projection defined by $(p, q) \mapsto (x_{i,j} + pu_{i,j}, y_{i,j} + pv_{i,j})$. Also, for each j, consider the map $\psi_{i,j} \colon \pi_i^{-1}(E_{i,j}) \to Y_{i,j}$ defined by $(p_1, q_1, p_2, q_2) \mapsto (p, q)$ where p is such that $(p_1, p_2) = (x_{i,j} + pu_{i,j}, y_{i,j} + pv_{i,j})$ and $q = u_{i,j}q_1 + v_{i,j}q_2$. In other words, the $S^1 \times S^1$ in Y_i above a point $(p_1, p_2) \in E_{i,j}$ collapses to the S^1 in $Y_{i,j}$ above (p_1, p_2), with the $(-v_{i,j}, u_{i,j})$-curves in $S^1 \times S^1$ being the curves that collapse to points. Note that on $\pi_i^{-1}(E_{i,j})$ we have $\pi_i = \pi_{i,j} \circ \psi_{i,j}$. Then there exists a unique symplectic 4-manifold (X_i, ω_i) equipped with the following smooth maps:

(1) A map $\mu_i\colon X_i \to \mathbb{R}^2$ with $\mu_i(X_i) = R_i$. (This is the "moment map" for a Hamiltonian torus action on (X_i, ω_i), after identifying $\mathbb{R}^2$ with the dual of the Lie algebra of $S^1 \times S^1$.)

(2) A map $\phi_i\colon Y_i \to X_i$ such that $\pi_i = \mu_i \circ \phi_i$ and such that ϕ_i is a symplectomorphism from $(\pi_i^{-1}(R_i \setminus \partial R_i), \eta_i)$ to $(\mu_i^{-1}(R_i \setminus \partial R_i), \omega_i)$. We think of this as giving us coordinates (p_1, q_1, p_2, q_2) on X_i with respect to which $\omega_i = dp_1 \wedge dq_1 + dp_2 \wedge dq_2$, except that the S^1-valued coordinates (q_1, q_2) degenerate along $\mu_i^{-1}(\partial R_i)$ in various ways determined by the next set of maps.

(3) A map $\phi_{i,j}\colon Y_{i,j} \to \mu_i^{-1}(E_{i,j})$, for each $j = 0, \ldots, n_i + 1$, such that $\pi_{i,j} = \mu_i \circ \phi_{i,j}$ and such that $\phi_{i,j}$ restricts to a symplectomorphism from $(\pi_{i,j}^{-1}(E_{i,j} \setminus \partial E_{i,j}), \eta_{i,j})$ to $(\mu_i^{-1}(E_{i,j} \setminus \partial E_{i,j}), \omega_i)$.

In addition, for each point $P_{i,j} \in R_i$, $\mu_i^{-1}(P_{i,j})$ is a single point. The following are consequences of this fact and the relationships amongst these maps:

(1) For each point $P \in R_i \setminus \partial R_i$, $\mu_i^{-1}(P)$ is a torus.

(2) For each point $P \in E_{i,j} \setminus \partial E_{i,j}$, $\mu_i^{-1}(P)$ is a circle.

(3) For each compact edge $E_{i,j}$, $j = 1, \ldots, n_i$, $\mu_i^{-1}(E_{i,j})$ is a symplectically embedded 2-sphere $S_{i,j}$ with symplectic area $2\pi\lambda_{i,j}$.

(4) For each of the two noncompact edges $E_{i,0}$ and E_{i,n_i+1}, $D_{i,j} = \mu_i^{-1}(E_{i,j})$ is a symplectically embedded open disk with symplectic area $2\pi\lambda_{i,j}$ ($j = 0$ or $n_i + 1$).

(5) Each point $\mu_i^{-1}(P_{i,j})$, for $j = 1, \ldots, n_i + 1$, is a point of ω-orthogonal intersection between $\mu_i^{-1}(E_{i,j-1})$ and $\mu_i^{-1}(E_{i,j})$.

(6) Given an embedded arc $\gamma\colon [0,1] \to R_i$ meeting ∂R_i transversely at one point $\gamma(1)$ in the interior of an edge $E_{i,j}$, the submanifold $\mu_i^{-1}(\gamma([0,1]))$ is a solid torus naturally parameterized using the coordinates (t, q_1, q_2) as $[0,1] \times S^1 \times S^1$, with $\{1\} \times S^1 \times S^1$ collapsed to S^1 so that the $(-v_{i,j}, u_{i,j})$ curves collapse to points.

(7) Each 2-sphere $S_{i,j} = \mu_i^{-1}(E_{i,j})$, for $j = 1, \ldots, n_i$, has self-intersection $S_{i,j} \cdot S_{i,j} = -\det(\tau_{i,j-1}, \tau_{i,j+1})$. This can be seen by using the preceding point to understand the topology of the S^1-bundle over S^2 which is the boundary of a tubular neighborhood of $S_{i,j}$. This bundle is the inverse image

under μ_i of a parallel copy of $E_{i,j}$, translated into the interior of R_i and extended until it intersects the two adjacent edges $E_{i,j-1}$ and $E_{i,j+1}$. Thus the bundle is the result of Dehn filling the two boundary components of $[0,1] \times S^1 \times S^1$ according to the rule in the preceding point, which gives an Euler class of $-\det(\tau_{i,j-1}, \tau_{i,j+1})$.

Furthermore, focusing attention on the end $\mu_i^{-1}((x_{i,0}, x_{i,0} + \epsilon) \times [y_{i,0}, y_{i,0} + \epsilon))$, which is diffeomorphic to $(x_{i,0}, x_{i,0} + \epsilon) \times S^1 \times D$, where D is an open disk in $\mathbb{R}^2$ of an appropriate radius, everything can be written explicitly as follows, using coordinates $t \in (x_{i,0}, x_{i,0}+\epsilon)$, $\alpha \in S^1$ and polar coordinates $(r, \theta) \in D$:

(1) $p_1 = t, q_1 = \alpha, p_2 = y_{i,0} + \frac{1}{2}r^2, q_2 = \theta$, so that $\mu_i(t, \alpha, r, \theta) = (t, y_{i,0} + \frac{1}{2}r^2)$.
(2) $\omega_i = dt \wedge d\alpha + rdr \wedge d\theta$.
(3) $f_i(t, \alpha, r, \theta) = \frac{1}{2}r^2$.

Finally, to get the desired convexity, the radial vector field on $\mathbb{R}^2$ radiating out from the origin lifts to a Liouville vector field V_i defined on $X_i \setminus \pi_i^{-1}(E_{i,0} \cup \cdots \cup E_{i,n})$ which is, in (p_1, q_1, p_2, q_2) coordinates, given by $V_i = p_1\partial_{p_1} + p_2\partial_{p_2}$. (The fact that V_i is defined on $\pi_i^{-1}(E_{i,n_i+1})$ comes from the fact that the line L_{i,n_i+1} passes through O, so that E_{i,n_i+1} is tangent to the radial vector field.) On the end $\mu_i^{-1}((x_{i,0}, x_{i,0} + \epsilon) \times [y_{i,0}, y_{i,0} + \epsilon)) \cong (x_{i,0}, x_{i,0}+\epsilon) \times S^1 \times D$, with coordinates (t, α, r, θ) as above, V_i is given by $V = t\partial_t + (\frac{1}{2}r + \frac{y_{i,0}}{r})\partial_r$. ■

Thus we see that $(X_i, \omega_i, C_i = \mu_i^{-1}(E_{i,0} \cup \cdots \cup E_{i,n_i}), f_i = g_i \circ \mu_i, V_i)$ is the desired neighbourhood 5-tuple for the legs and that, furthermore, we have everything written down explicitly in local coordinates on the end of X_i which is a neighbourhood of the disk $D_{i,0}$.

Now we construct the neighbourhood 5-tuple for the central vertex using the following lemma (which we have stated in a form which allows also for positive genus, in case it should ever be useful):

Lemma 2.4. Suppose that we are given a compact connected surface Σ with $m > 0$ boundary components $\partial_1\Sigma, \ldots, \partial_m\Sigma$, $k \geq 0$ negative real numbers $c_1, \ldots, c_k$, and $m-k > 0$ positive real numbers $c_{k+1}, \ldots, c_m$, with $c_1 + \cdots + c_k + c_{k+1} + \cdots + c_m > 0$. Then there exists a symplectic form β on Σ and a Liouville vector field W defined on all of Σ, pointing in along $\partial_i\Sigma$ for $1 \leq i \leq k$ and pointing out along $\partial_i\Sigma$ for $k+1 \leq i \leq m$, such that, for each $i = 1, \ldots, m$, a collar neighbourhood of $\partial_i\Sigma$ can be parameterized as $(c_i - \epsilon, c_i] \times S^1$ with coordinates (t, α) with respect to which $\beta = dt \wedge d\alpha$ and $W = t\partial_t$. □

Proof. Note that the condition that $c_1 + \cdots + c_m > 0$ is a necessary condition because $2\pi(c_1 + \cdots + c_m) = \int_{\partial\Sigma} t d\alpha = \int_{\partial\Sigma} \imath_W \beta = \int_\Sigma \beta > 0$. There are probably numerous ways to see that this lemma is true; in fact we need only extend the 1-forms $td\alpha$ on each end to a 1-form γ on all of Σ such that $d\gamma > 0$. Our idea here is to apply Weinstein's techniques [33] for attaching symplectic handles to symplectic manifolds with convex boundary in the relatively trivial case of dimension 2; in this case the "contact forms" on the 1-dimensional boundary components are nowhere zero 1-forms. Once we get the 1-forms on the boundary correct, then flowing in along the Liouville vector fields produces the correct parameterization of the ends.

First we claim that the lemma is true if $m - k = 1$, in which case the vector field should point out along only one boundary component. Start with the disjoint union $\amalg_{i=1}^{k}([c_i - \epsilon, c_i] \times S^1, dt \wedge d\alpha)$ of k symplectic cylinders with the Liouville vector field $t\partial_t$ (or a very small disk if $k = 0$ with the radial vector field) and attach Weinstein's 1-handles to the boundaries along which the Liouville vector fields point out, to make a symplectic surface diffeomorphic to Σ, with a Liouville vector field with the correct behavior along $\partial_1\Sigma, \dots, \partial_k\Sigma$ and pointing out along $\partial_m\Sigma$. Now simply attach a symplectization collar to $\partial_m\Sigma$ if $\int_{\partial_m\Sigma} \iota_W\beta$ is too small, or remove a collar (by flowing backwards along W) if it is too big, and thus achieve the correct behavior along $\partial_m\Sigma$.

Now if $m - k > 1$, we can first build a symplectic surface (Σ', β') with the same genus as Σ and $k+1$ boundary components, with a Liouville vector field W, with k boundary components along which W points in and the behavior is correct, and one boundary component ∂_+ along which the vector field points out, arranging that $\int_{\partial_+} \iota_{W'}\beta' < 2\pi(c_{k+1} + \cdots c_m)$. Then use the same ideas as in the preceding paragraph to build a $(m - k+1)$-punctured sphere with one boundary component ∂_- along which the Liouville vector field points in, so as to be able to glue to ∂_+, and with $m - k$ components along which the vector field points out with the correct behavior. Then glue the two pieces together.

The neighbourhood 5-tuple needed for the central vertex is now ($X_0 = \Sigma \times D$, $\omega_0 = \beta + \frac{1}{2\pi} r dr \wedge d\theta, C_0 = \Sigma \times \{(0,0)\}, f_0 = \frac{1}{2}r^2, V_0 = W + (\frac{1}{2}r + \frac{y_0}{r})\partial_r$). Because $y_0 = y_{i,0}$ for each $i = 1, \dots, m$, if we implement the above lemma using $c_i = x_{i,0} + \delta$ for some small $\delta > 0$, and for Σ an m-punctured sphere, we will be able to glue the 5-tuples together. The area of the central sphere S_0 will then be $2\pi(x_{1,1} + x_{2,1} + \cdots + x_{m,1})$ by Stokes' theorem.

It remains to compute $S_0 \cdot S_0$. To do this note that, for some small $\delta > 0$, the submanifold N of X defined by

$$N = (\Sigma \times D_{\sqrt{2\delta}}) \cup \bigcup_{i=1}^{m} \mu_i^{-1}(R_i \cap ((x_{i,0}, \infty) \times [y_{i,0}, y_{i,0} + \delta)))$$

(where $D_{\sqrt{2\delta}}$ is an open disk of radius $\sqrt{2\delta}$) is a tubular neighbourhood of S_0. Then we can see ∂N as diffeomorphic to $\Sigma \times S^1$ with the i'th component $(\partial_i \Sigma) \times S^1$ of $\partial(\Sigma \times S^1)$ filled in with a solid torus so that the $(-v_{i,1}, u_{i,1})$ curves in $(\partial_i \Sigma) \times S^1 = S^1 \times S^1$ bound disks. But since $v_{i,1} = 1$ this means that the $(1, -u_{i,1})$ curves are filled, so this 3-manifold is the S^1-bundle over S^2 of Euler class $-u_{1,1} - u_{2,1} - \cdots - u_{m,1}$. Since the base S^2 in this case is precisely the central vertex sphere S_0, this tells us that $S_0 \cdot S_0 = -u_{1,1} - \cdots - u_{m,1}$. ■

Now we need to translate Proposition 2.3 into a proof of Theorem 2.2.

Proof of Theorem 2.2. Given a star-shaped plumbing graph Γ, label the spheres $S_0, S_{i,j}$, $1 \le i \le m, 1 \le j \le n_i$, so that S_0 corresponds to the central m-valent vertex of Γ and $S_{i,1}, \ldots, S_{i,n_i}$ correspond in order to the vertices on the i'th leg, with $S_0 \cdot S_{i,1} = 1$. Let the corresponding self-intersections be $s_0, s_{i,j}$ and let the areas be $a_0, a_{i,j}$. We need to produce the points $P_{i,j}$, $1 \le i \le m, 0 \le j \le n_i+2$ satisfying the conditions of Proposition 2.3, giving the desired self-intersections and areas.

We first consider the vectors $\tau_{i,j} \in \mathbb{Z}^2$, $1 \le i \le m, 0 \le j \le n_i + 1$. We are forced to have $\tau_{i,0} = (1,0)^T$ and $\tau_{i,1} = (u_{i,1}, 1)^T$. We can make a choice for the values of $u_{i,1}$ so long as $u_{1,1} + \cdots + u_{m,1} = -s_0$, so that $S_0 \cdot S_0 = -u_{i,1} - \ldots - u_{m,1} = s_0$. Now note that each $\tau_{i,j+1}$ for $j = 1, \ldots, n_i$ is completely determined by $\tau_{i,j-1}$ and $\tau_{i,j}$ and the constraints that $\det(\tau_{i,j-1}, \tau_{i,j+1}) = -s_{i,j}$ and $\det(\tau_{i,j}, \tau_{i,j+1}) = +1$, so that our choices for $\tau_{i,0}$ and $\tau_{i,1}$ determine $\tau_{i,2}, \ldots, \tau_{i,n_i+1}$. In fact, $\tau_{i,j+1} = -\tau_{i,j-1} - s_{i,j}\tau_{i,j}$.

For each $i = 1, \ldots, m$, let $\sigma_i = u_{i,n_i+1}/v_{i,n_i+1}$, the reciprocal of the slope of the terminal vector τ_{i,n_i+1}. We claim that if Γ is negative definite then, for any choice of $u_{1,1}, u_{2,1}, \ldots, u_{m,1}$ such that $u_{1,1} + u_{2,1} + \cdots + u_{m,1} = -s_0$, we will automatically have $\sigma_1 + \cdots + \sigma_m > 0$. Given this claim, the existence of suitable points $P_{i,j}$ can be seen as follows:

Having determined the vectors $\tau_{i,j}$, the points $P_{i,j}$ for $i = 1, \ldots, m, j = 2, \ldots, n_i + 1$ are completed determined by the $P_{i,1}$'s and the areas $a_{i,j}$. Furthermore, the fact that P_{i,n_i+1} must lie on the line L_{i,n_i+1} passing through the origin tangent to the vector τ_{i,n_i+1} constrains $P_{i,1}$ to lie on a particular line parallel to τ_{i,n_i+1}; i.e. there is some constant K_i, determined by the $\tau_{i,j}$'s and the $a_{i,j}$'s, such that $x_{i,1} = \sigma_i y_{i,1} + K_i$. Because the points $P_{i,j}$ must all lie to the left of the line L_{i,n_i+1}, which passes through the origin, we know that $K_i < 0$. The precise location of P_{i,n_i+2} on L_{i,n_i+1} is not important, so we ignore it. The precise location of $P_{i,0}$ is also not important, provided it is close enough to $P_{i,1}$ and provided $x_{1,1} + x_{2,1} + \cdots + x_{m,1} > 0$. Recall that we have the additional constraint that $y_{i,1} = y_{i,0} = y_0$ for some y_0; thus we can think of each $x_{i,1}$ as determined by a linear function of y_0: $x_{i,1} = \sigma_i y_0 + K_i$. We must now choose y_0 appropriately so that

$2\pi(x_{1,1} + x_{2,1} + \cdots + x_{m,1}) = a_0$, the given area of S_0. We have $x_{1,1} + x_{2,1} + \cdots + x_{m,1} = (\sigma_1 + \cdots + \sigma_m)y_0 + (K_1 + \cdots + K_m)$. But since $\sigma_1 + \cdots + \sigma_m > 0$ and $K_1 + \cdots + K_m < 0$, we can realize any given positive value for $x_{1,1} + \cdots + x_{m,1}$ by choosing an appropriate $y_0 > 0$.

Now to prove the claim, we first appeal to [18, Theorem 5.2] stating that a star-shaped graph G is negative definite if and only if $s_0 + r_1 + \cdots + r_m < 0$, where r_i is the negative reciprocal of the continued fraction corresponding to the i'th leg, i.e.

$$r_i = -\frac{1}{s_{i,1} - \frac{1}{s_{i,2} - \cdots - \frac{1}{s_{i,n_i}}}} \tag{2.1}$$

In the proof of [32, Theorem 9.20] Symington shows that if $\tau_0, \ldots, \tau_{n+1}$, $\tau_i = (u_i, v_i)^T$, is a list of vectors in $\mathbb{Z}^2$ with $\tau_0 = (0, -1)^T$, $\tau_1 = (1, 0)^T$, $\det(\tau_i, \tau_{i+1}) = 1$ and $\det(\tau_{i-1}, \tau_{i+1}) = b_i$, then the reciprocal slope σ of the last vector is given by

$$\sigma = \frac{u_{n+1}}{v_{n+1}} = b_1 - \frac{1}{b_2 - \cdots - \frac{1}{b_n}}$$

We can change such a list to one where $\tau_0 = (1, 0)^T$ and $\tau_1 = (u, 1)^T$ for some given $u \in \mathbb{Z}$ by applying the linear transformation

$$\tau_i \mapsto \begin{pmatrix} u & -1 \\ 1 & 0 \end{pmatrix} \tau_i$$

This transforms σ to $u - 1/\sigma$. In other words, after applying this transformation, σ is now given by:

$$\sigma = u - \frac{1}{b_1 - \frac{1}{b_2 - \cdots - \frac{1}{b_n}}}$$

Applying this to each leg of our star-shaped graph, with the initial condition $\tau_{i,1} = (u_{i,1}, 1)^T$ and $\det(\tau_{i,j-1}, \tau_{i,j+1}) = -s_{i,j}$ we get that

$$\sigma_i = u_{i,1} + \frac{1}{s_{i,1} - \frac{1}{s_{i,2} - \cdots - \frac{1}{s_{i,n_i}}}} = u_{i,1} - r_i$$

Thus $\sigma_1 + \cdots + \sigma_m = -(s_0 + r_1 + \cdots + r_m)$ and the claim is proved. ∎

Remark 2.5. Since Lemma 2.4 allows for any genus, Theorem 2.1 is actually true even when the surface corresponding to the central vertex has positive genus. It seems that a similar picture provides ω-convex neighbourhood for any collection of symplectic spheres intersecting each other according to a *negative definite* plumbing tree, i.e. the assumption of Theorem 2.2 on Γ being star-shaped can be removed. We hope to return to this question in a future project.

3 Tight contact structures on certain Seifert fibered 3-manifolds

In order to find the right gluing map for performing the rational blow-down process, we will invoke a classification result of tight contact structures on certain small Seifert fibered 3-manifolds. Let us start with some generalities.

As is costumary, we say that a 3-manifold Y is a *small Seifert fibered space* if it can be given by the surgery diagram of Figure 9. Here we assume that $s_0 \in \mathbb{Z}$ and $r_i \in (0,1) \cap \mathbb{Q}$ (then $(s_0; r_1, r_2, r_3)$ are the *normalized Seifert invariants* of $M = M(s_0; r_1, r_2, r_3)$). By applying the inverse slam dunk operation (cf. [11, Figure 5.30]), the diagram of Figure 9 can be easily transformed into a star-shaped plumbing as it is given by Figure 10. The plumbing coefficients $[a^i_1, \ldots, a^i_{k_i}]$ on the i^{th} leg (satisfying $a^i_j \in \mathbb{Z}$ and $a^i_j \leq -2$) are specified by the continued fraction coefficients of the rational number $-\frac{1}{r_i} < -1$ (cf. Equation (2.1)). Notice that all plumbing graphs in the Introduction (Figures 1–7) give rise to 4-manifolds with small Seifert fibered 3-manifold boundary.

In a remarkable series of papers [20, 21] Ozsváth and Szabó introduced an invariant for spin^c 3-manifolds: the (mod 2) *Ozsváth-Szabó homology group* $\widehat{HF}(Y, \mathbf{t})$ of a closed spin^c 3-manifold $(Y, \mathbf{t})$ is a finite dimensional vector space over the field $\mathbb{Z}_2$ of two elements. For a rational homology sphere the dimension of this vector space is odd for every spin^c structure. We say that a rational homology sphere Y is an *L-space* if

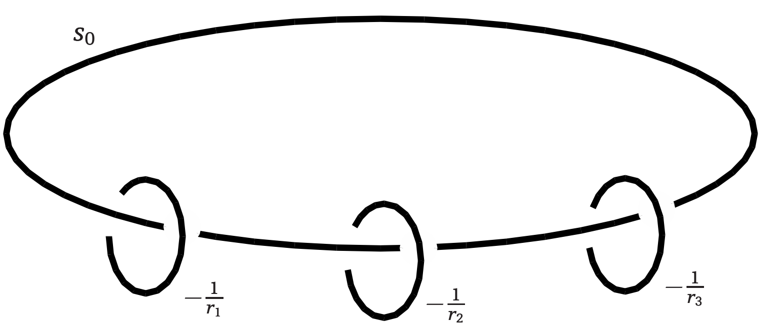

Figure 9 Surgery diagram for the Seifert fibered 3-manifold $M(s_0; r_1, r_2, r_3)$.

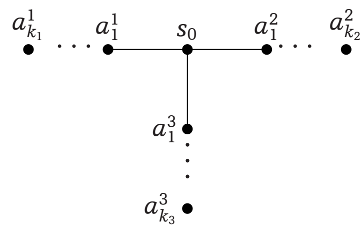

Figure 10 Plumbing diagram of a 4-manifold with boundary $M(s_0; r_1, r_2, r_3)$.

$\widehat{HF}(Y, \mathbf{t}) = \mathbb{Z}_2$ for every spin^c structure $\mathbf{t} \in Spin^c(Y)$, that is, Y has the simplest possible Ozsváth-Szabó homologies.

Recall that a contact structure ξ on any 3-manifold naturally induces a spin^c structure, which we will denote by $\mathbf{t}_\xi$. In addition, (Y, ξ) also gives rise to an element $c(Y, \xi) \in \widehat{HF}(-Y, \mathbf{t}_\xi)$, the *contact invariant* of (Y, ξ). (For the definition and basic properties of $c(Y, \xi)$ see [23, 14].)

As the next result (a compilation of theorems of Wu, Ghiggini and Plamenevskaya) shows, tight contact structures on small Seifert fibered 3-manifolds with simple Ozsváth-Szabó homologies (and $s_0 \leq -2$) admit a simple classification scheme. More precisely

Theorem 3.1. Suppose that the small Seifert fibered 3-manifold $M = M(s_0; r_1, r_2, r_3)$ satisfies $s_0 \leq -2$ and M is an L-space. Then two tight contact structures ξ_1, ξ_2 on M are isotopic if and only if $\mathbf{t}_{\xi_1} = \mathbf{t}_{\xi_2}$. □

Proof. One direction of the statement is obvious: isotopic contact structures are homotopic as 2-plane fields, hence induce the same spin^c structure. The converse, however, is more subtle, since it states that in these 3-manifolds the spin^c structure (which is a homotopic invariant of the contact structure) determines the tight contact structure up to isotopy.

Let Γ denote the star-shaped plumbing graph we get from the surgery diagram of $M = M(s_0; r_1, r_2, r_3)$ by inverse slam dunks. The corresponding 4-manifold Z_Γ is then given by a sequence of 2-handle attachments along the unknots corresponding to the vertices of Γ. Working with a projection now it is easy to see that each unknot can be isotoped until it becomes the Legendrian unknot, i.e., a Legendrian knot isotopic to the unknot with Thurston-Bennequin invariant -1. The assumption $s_0 \leq -2$ ensures that by adding sufficiently many zig-zags to these Legendrian unknots we end up with a Legendrian link on which contact (-1)-surgery yields Z_Γ together with a Stein structure on it [1, 10]. Notice, however, that there is a certain freedom in adding the zig-zags to the unknots:

moving a zig-zag from left to right (or from right to left) will change the rotation number of the particular knot, which in turn will change the first Chern class of the resulting Stein structure on Z_Γ. According to a result of Lisca and Matić [13] Stein structures on a fixed 4-manifold with different first Chern classes induce nonisotopic tight contact structures on the boundary of the 4-manifold. The number of tight contact structures one can distinguish in this way can be easily computed from the framing coefficients of the graph Γ. Furthermore, it was shown by Plamenevskaya [24] that if two Stein structures on a 4-manifold induce two contact structures ξ_1, ξ_2, and for the two Stein structures we have $c_1(J_1) \neq c_1(J_2)$ then for the (mod 2 reduced) contact Ozsváth-Szabó invariants we have $c(M, \xi_1) \neq c(M, \xi_2)$. In summary, the Legendrian surgery construction described above allowed us to construct a finite set of Stein fillable (hence tight) contact structures on M (corresponding to the different choices of left and right zig-zags), and all these structures have different (and nonzero) contact Ozsváth-Szabó invariants.

Now we are ready to prove the theorem. Using convex surface techniques, it was shown by Wu [34] that for a small Seifert 3-manifold with $s_0 \leq -3$ the above set of Stein fillable contact structures, in fact, contains *all* tight contact structures on the 3-manifold at hand. (For these 3-manifolds the L-space condition is automatically satisfied, cf. Lemma 4.1.) For the remaining $s_0 = -2$ case we argue as follows: According to [15] the assumption for M being an L-space implies that $-M$ admits no transverse contact structure. In [7] this property is translated to a numerical condition for r_1, r_2, r_3, which in turn, implies that the number of tight contact structures on M is bounded above by the number of Stein fillable contact structures constructed by Legendrian surgery along the Legendrian links described above. In short, under the assumption of the theorem, the Legendrian surgery construction described at the beginning of the proof produces all tight contact structures on M.

Appealing to the L-space property again, now we can easily conclude the proof: if ξ_1, ξ_2 are nonisotopic tight contact structures on M then according to the above said we have $c(M, \xi_1) \neq c(M, \xi_2)$. Assuming that for the induced spin^c structures $\mathbf{t}_{\xi_1} = \mathbf{t}_{\xi_2} = \mathbf{t}$ holds we easily reach a contradiction: in this case the contact invariants $c(M, \xi_1), c(M, \xi_2) \in \widehat{HF}(-M, \mathbf{t}) = \mathbb{Z}_2$ are distinct, nonzero elements, which is clearly impossible. ■

Remark 3.2. In fact, a very similar proof (together with the classification of tight contact structures on small Seifert fibered 3-manifolds with $s_0 \geq 0$ [8]) applies for $s_0 \geq 0$ as well. (It is worth mentioning that if $s_0(M) \geq 0$ then $s_0(-M) \leq -3$, therefore any M

with $s_0(M) \geq 0$ is still an L-space.) The only difference is in the construction of the appropriate 4-manifold (carrying the Stein structures): for $s_0 \geq 0$ we need to introduce an appropriate Stein 1-handle as well, cf. the argument of [8]. We will not use this fact in the present paper. A similar statement (that is, that tight contact structures with isomorphic spin^c structures are isotopic) is expected in the case when $s_0 = -1$ and the 3-manifold is an L-space; for related discussion and partial results see [9]. Most probably the same statement also holds for *strongly fillable* contact structures on boundaries of negative definite plumbings which are L-spaces. Considering all tight contact structures, such a statement cannot be true, since many of these 3-manifolds are toroidal, hence contain infinite families of homotopic, nonisomorphic tight contact structures. (These infinite families are constructed by inserting Giroux torsions, resulting in contact structures which are not strongly fillable [6].) For strongly fillable structures, however, one can expect that the strong filling can be chosen to be diffeomorphic to the plumbing 4-manifold, and then an adaptation of the proof above would provide the same result.

4 The proof of Theorem 1.2

We start this section by proving that the result of Theorem 3.1 applies for the plumbing manifolds given by graphs in $\mathcal{G}$.

Lemma 4.1. Suppose that $\Gamma \in \mathcal{G}$. Then the star-shaped graph Γ is negative definite and the 3-manifold ∂M_Γ is an L-space. □

Proof. The plumbing graph $\Gamma \in \mathcal{G}$ is negative definite since it embeds in a negative definite lattice [29]. (Alternatively, a simple computation and the application of [18, Theorem 5.2] shows the same.) There are many ways to verify that ∂M_Γ is an L-space. The direct application of the algorithm of [22] (which applies for negative definite plumbing graphs with at most one "bad" vertex) implies the result at once, although this proof involves some computations. (In fact, for graphs in $\mathcal{W} \cup \mathcal{N}$ the result is stated in [22], since these graphs involve no "bad" vertices.) Alternatively, we can argue as follows: as [29, Examples 8.3 and 8.4] show, the normal surface singularity defined by $\Gamma \in \mathcal{G}$ admits a rational homology disk smoothing and this property implies that the surface singularity is rational, cf. [29, Proposition 2.3]. (In yet another way, Laufer's algorithm can be easily applied to verify rationality of the surface singularities defined by graphs in $\mathcal{G}$.) The computation presented in [17] shows that the link of a rational singularity is an L-space, concluding our argument. ■

After all these preparations, now we are ready to turn to the proof of Theorem 1.2:

Proof of Theorem 1.2. Suppose that the S_i are symplectic spheres in the symplectic 4-manifold (X, ω), intersecting each other ω-orthogonally, and according to the plumbing tree $\Gamma \in \mathcal{G}$. By Theorem 2.1 there is a neighbourhood S_Γ of $\cup S_i$ such that its complement is a strong concave filling of its contact boundary. The contact structure on the boundary ∂S_Γ induces the spin^c structure which extends to S_Γ as $\mathbf{s}$ with the property that $c_1(\mathbf{s})$ satisfies the adjunction equality for all S_i (since the S_i are all smooth symplectic submanifolds).

Now consider the normal surface singularity given by Γ. As such, it defines a contact structure on its link, which induces the spin^c structure $\mathbf{t}$. Resolving the singularity doesn't change this contact structure (and hence the spin^c structure $\mathbf{t}$), but provides a 4-manifold diffeomorphic to S_Γ, with a spin^c structure $\mathbf{s}_{can}$. Since the vertices of Γ in the resolution correspond to complex curves, the first Chern class of $\mathbf{s}_{can}$ also satisfies the adjunction equality. This shows that $c_1(\mathbf{s}) = c_1(\mathbf{s}_{can})$, and since S_Γ is a simply connected 4-manifold, we conclude that $\mathbf{s} = \mathbf{s}_{can}$. This fact, however, identifies the spin^c structures of the contact structures on ∂S_Γ, which in the light of Theorem 3.1 implies that the contact structure on ∂S_Γ induced by the Liouville vector field is isotopic to the one induced on the link of the singularity. This last observation verifies the existence of the desired gluing contactomorphism, hence the symplectic gluing can be performed as in [3], concluding the proof. ■

Acknowledgments

The material is based upon work supported by the National Research Foundation of South Africa under Grant number 62124 (Hungarian Grant Number ZA-15/2006). The second author was also partially supported by OTKA T49449 and by the EU Marie Curie TOK grant BudAlgGeo. We would also like to thank the referee for helpful comments which have improved the exposition.

References

[1] Eliashberg, Y. "Topological Characterization of Stein Manifolds of Dimension > 2." *International Journal of Mathematics* 1 (1990): 29–46.

[2] ———. "On Symplectic Manifolds with some Contact Properties." *Journal of Differential Geometry* 33 (1991): 233–238.

[3] Etnyre, J. *Symplectic, Convexity in Low-Dimensional Topology (Athens, A, 1996), Topology and its Applications* 88 (1998): 3–25.

[4] Fintushel, R., and R. Stern. "Rational Blowdowns of Smooth 4-Manifolds." *Journal of Differential Geometry* 46 (1997): 181–235.

[5] ———. "Double node Neighborhoods and Families of Simply Connected 4-Manifolds with $b_2^+ = 1$." *Journal of the American Mathematical Society* 19 (2006): 171–180.

[6] Gay, D. "Four-Dimensional Symplectic Cobordisms Containing Three-Handles." *Geometry and Topology* 10 (2006): 1749–1760.

[7] Ghiggini, P. On Tight Contact Structures with Negative Maximal Twisting on Small Seifert Manifolds (2007): in preparation.

[8] Ghiggini, P., P. Lisca, and A. Stipsicz. "Classification of Tight Contact Structures on Small Seifert 3-Manifolds with $e_0 \geq 0$." *Proceedings of the American Mathematical Society* 134 (2006): 909–916.

[9] ———. "Tight Contact Structures on some Small Seifert Fibered 3-Manifolds." *American Journal of Mathematics* (forthcoming).

[10] Gompf, R. "Handlebody construction of Stein Surfaces." *Annals of Mathematics* 148 (1998): 619–693.

[11] Gompf, R., and A. Stipsicz. "4-Manifolds and Kirby Calculus." *Graduate Studies in Mathematics* 20 , AMS 1999.

[12] Grauert, H. "Über Modifikationen und exzepotionelle analytische Mengen." *Mathematische Annalen* 146 (1962): 331–368.

[13] Lisca, P., and G. Matić. "Tight Contact Structures and Seiberg-Witten Invariants." *Inventiones Mathematicae* 129 (1997): 509–525.

[14] Lisca, P., and A. Stipsicz. "Ozsváth-Szabó Invariants and Tight Contact 3-Manifolds, I." *Geometry and Topology* 8 (2004): 925–945.

[15] ———. "Ozsv áth-Szabó invariants and tight contact 3-manifolds, III." *Journal of Symplectic Geometry*, to appear arXiv:math.SG/0505493.

[16] Michalogiorgaki, M. "Rational Blow-Down along Wahl type Plumbing Trees of Spheres." *Algebraic and Geometric Topology*, to appear arXiv:math.GT/0607608.

[17] Némethi, A. "On the Ozsváth-Szabó Invariant of Negative Definite Plumbed 3-Manifolds." *Geometry and Topology* 9 (2005): 991–1042.

[18] Neumann, W., and F. Raymond. "Seifert manifolds, plumbing, μ-invariant and orientation reversing maps." In *Algebraic and Geometric Topology (Proceedings of the Symposium, University of California, Santa Barbara, California, 1977), Lecture Notes in Mathematics 664* Berlin: Springer, (1978): 163–196.

[19] Ohta, H., and K. Ono. "Simple Singularities and Topology of Symplectically Filling 4-Manifold." *Commentarii Mathematici Helvetici* 74 (1999): 575–590.

[20] Ozsváth, P., and Z. Szabó. "Holomorphic Disks and Topological Invariants for Closed Three-Manifolds." *Annals of Mathematics* 159 (2004): 1027–1158.

[21] ———. "Holomorphic Disks and Three-Manifold Invariants: Properties and Applications." *Annals of Mathematics* 159 (2004): 1159–1245.

[22] ———. "On the Floer Homology of Plumbed Three-Manifolds." *Geometry and Topology* 7 (2003): 185–224.

[23] ———. "Heegaard Floer Homologies and Contact Structures." *Duke Mathematical Journal* 129 (2005): 39–61.

[24] Plamenevskaya, O. "Contact Structures with Distinct Heegaard Floer Invariants." *Mathematical Research Letters* 11 (2004): 547–561.

[25] Park, J. "Seiberg-Witten Invariants of Generalized Rational Blow-Downs." *Bulletin of the Australian Mathematical Society* 56 (1997): 363–384.

[26] ———. "Simply Connected Symplectic 4-Manifolds with $b_2^+ = 1$ and $c_1^2 = 2$." *Inventiones Mathematicae* 159 (2005): 657–667.

[27] Park, J., A. Stipsicz, and Z. Szabó. "Exotic Smooth Structures on $\mathbb{CP}^2\#5\overline{\mathbb{CP}^2}$." *Mathematical Research Letters* 12 (2005): 701–712.

[28] Stipsicz, A., and Z. Szabó. "An Exotic Smooth Structure on $\mathbb{CP}^2\#6\overline{\mathbb{CP}^2}$." *Geometry and Topology* 9 (2005): 813–832.

[29] Stipsicz, A., Z. Szabó, and J. Wahl. "Rational Blow-Downs and Smoothings of Surface Singularities." *Geometric Topology*, to appear arXiv:math.SG/0611157.

[30] Symington, M. "Symplectic Rational Blowdowns." *Journal of Differential Geometry* 50 (1998): 505–518.

[31] ———. "Generalized Symplectic Rational Blowdowns." *Algebraic and Geometric Topology* 1 (2001): 503–518.

[32] ———. "Four dimensions from two in symplectic topology." In *Topology and Geometry of Manifolds (Athens, GA, 2001), Proceedings of the Symposium Pure Mathematics, 71* American Mathematics Society: Providence, RI, (2003): 153–208.

[33] Weinstein, A. "Contact Surgery and Symplectic Handlebodies." *Hokkaido Mathematical Journal* 20, no. 2 (1991): 241–251.

[34] Wu, H. "Legendrian Vertical Circles in Small Seifert Spaces." *Communications in Contemporary Mathematics* 8 (2006): 219–246.